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Action, hamiltonian and CFT for $2D$ black holes

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Abstract: The boundary terms in the hamiltonian, in the presence of horizons, are carefully analyzed in a simple $2D$ theory admitting AdS black holes. The agreement between the euclidean, Gibbons-Hawking approach and CFT through Cardy's formula is obtained modulo certain assumptions regarding the spectrum of the Virasoro's algebra. There is no discrepancy factor $\sqrt{2}$ once the appropriate boundary conditions are properly recognized. The results obtained here are of general validity, since they rely on general properties of black holes. In particular, the central charge can be understood as a classical result without invoking a string or any other microscopic theory. The peculiar feature of gravity, that the on-shell hamiltonian is determined by boundary terms, is the reason of the mentioned agreement.

Keywords: Space-Time Symmetries, 2D Gravity, Black Holes.

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Contents

1. Introduction

Gravity in $2D$ is a very rich and much investigated subject (a review of early work are the TASI lectures by Ginsparg-Moore [1]). Under the spell of modern string theory, there has been recently a lot of activity aiming to understood the subtle aspects of AdS/CFT correspondence in $2D$ gravity [2]–[8] (even two-dimensional de Sitter space has been understood as a variant of AdS/CFT correspondence [9]).

A consistent piece of work in $2D$ gravity has always been to understand AdS_2 black holes [10]–[14]. Most researches almost invariably focused on the asymptotic symmetries of AdS_2 gravitating system. It seems instructive, and to some extent important, to look instead at the near horizon symmetries, and the purpose of this letter is to do this in the hamiltonian framework of AdS gravity (see [15] for applications to AdS/CFT correspondence). We first determine the regularity (as opposed to boundary) conditions near the horizon. We then proceed to find the corresponding hamiltonian. A new term at the horizon will be found, which can be interpreted as a shift in energy due to a central charge in the diffeomorphism algebra. This will be used to evaluate the entropy, which will then be compared with the standard thermodynamics of black holes, finding complete agreement (no $\sqrt{2}$ discrepancy [13]).

2. Action and hamiltonian

The theory we will consider is the one defined by the Jackiw-Teitelboim's action [16]

$$I = \frac{1}{2}\int |g|^{1/2}\eta(R + 2\lambda^2) \tag{2.1}$$

which leads to the equations of motion

$$R + 2\lambda^2 = 0\,, \qquad g_{ab}\Box\eta - \nabla_a\nabla_b\eta = \lambda^2 g_{ab}\eta\,. \tag{2.2}$$

Black holes in this model have been studied in a number of papers [10, 11, 14, 17, 18]. The standard black hole solution is the linear dilaton vacuum, with metric

$$ds^2 = -(\lambda^2 x^2 - a^2)dt^2 + (\lambda^2 x^2 - a^2)^{-1} dx^2 \tag{2.3}$$

and dilaton $\eta = \eta_0 \lambda x$, where a and η_0 are integration constants. The event horizon is located at $x = x_0 = \lambda^{-1} a$. Given that the solution is uniquely determined by (i), existence of a timelike U(1) isometry and (ii), asimptotically AdS behaviour, one may wonder why the black hole should exhibit any thermodynamics property at all.

To better understanding this point, we are going to discuss how big is the phase space of the black hole, and to this aim we need the hamiltonian. We start then with a general metric

$$ds^2 = -N^2 dt^2 + \sigma^2 (dx + V dt)^2 \,, \qquad |g|^{1/2} = N\sigma \tag{2.4}$$

and define $U = -\dot{\sigma} + (\sigma V)'$, where a prime means derivative w.r.t. x. A calculation shows that

$$\begin{aligned} I = I(\text{bulk}) - \int dt\, \eta(\sigma^{-1} N' - N^{-1} V U)_X - \left[\int_{x_0}^{X} dx \eta N^{-1} U \right]_{t_2}^{t_1} + \\ + \lim_{x \to x_0} \int dt \left(\eta \sigma^{-1} N' - N^{-1} \eta V U \right) \end{aligned} \tag{2.5}$$

with

$$\begin{aligned} I(\text{bulk}) = -\int_{x_0}^{X} \left(\frac{\dot{\eta}\dot{\sigma}}{N} - \frac{\eta' N'}{\sigma} + \frac{V (V\sigma)' \eta'}{N} - \right. \\ \left. - \frac{\dot{\sigma} V \eta'}{N} - \frac{\dot{\eta} (\sigma V)'}{N} - \lambda^2 N \sigma \eta \right) d^2 x \,. \end{aligned} \tag{2.6}$$

Of course, in doing so one is concerned with the evolution in the exterior region, marked by $X > x > x_0 = a\lambda^{-1}$, where X will be taken to infinity when necessary. To simplify things let us set $V = 0$ from now on. We also omit the limits of spatial integration, leaving it understood that the range is actually $x \in [x_0, X]$. Under a small change of the metric and dilaton, the action has a variation

$$\begin{aligned} \delta I = \text{“terms giving equations of motion”} - \int dx \left[N^{-1} \dot{\eta} \delta\sigma \right]_1^2 + \\ + \int dt \left[\sigma^{-1} \eta' \delta N \right]_{x_0}^{X} - \int dt \left[\eta \delta \left(\frac{N'}{\sigma} \right) \right]_{x_0}^{X} + \int dx \left[\eta \delta \left(\frac{\dot{\sigma}}{N} \right) \right]_1^2 . \end{aligned} \tag{2.7}$$

Before going any further, a crucial issue will be to decide what kind of regularity conditions have to be imposed near the black hole horizon (see [19] for a discussion in four dimensions). This is the place where the U(1) isometry of time translation has a fixed point. We also require that near the horizon the geometry be isometric to a flat disk, so that the Euler characteristic will be $\chi = 1$. Both conditions can be met by[1]

$$N(x_0) = 0\,, \qquad (\sigma^{-1}N')_{|x=x_0} = \kappa\,, \tag{2.8}$$

where κ is the surface gravity of the black hole. We also needs to fix the metric, the lapse and η at t_1, t_2 and infinity.[2] Looking at (2.7), we see that we have to refine the action to read

$$I_{\rm NEW} = I + \int dt\,(\sigma^{-1}\eta N')_{|X} - \int dx\,\left[N^{-1}\eta\dot{\sigma}\right]_1^2 . \tag{2.9}$$

Even without bothering about boundary terms, we can write a canonical hamiltonian associated with I.[3] After standard manipulations, we obtain

$$\begin{aligned} H = & \int dx\, N\left[-P_\sigma P_\eta + \left(\frac{\eta'}{\sigma}\right)' - \lambda^2\sigma\eta\right] - \left(\frac{\eta' N}{\sigma}\right)_{|X} + \\ & + \left(\frac{\eta' N}{\sigma}\right)_{|x_0} + \left(\frac{\eta N'}{\sigma}\right)_{|X} - \left(\frac{\eta N'}{\sigma}\right)_{|x_0}, \end{aligned} \tag{2.10}$$

where $P_\sigma = -N^{-1}\dot{\eta}$, $P_\eta = -N^{-1}\dot{\sigma}$ and $P_N = 0$. The first two boundary terms have appeared because we wrote the hamiltonian in a form that displays the constraint

$$\mathcal{H}_\perp = -P_\sigma P_\eta + \left(\frac{\eta'}{\sigma}\right)' - \lambda^2\sigma\eta\,. \tag{2.11}$$

The last term in (2.10) is the crucial one, that will be related to a Virasoro's central charge. It is clear that it exists only because we wanted to predict the outer region of the black hole. The hamiltonian on a geodesically complete spatial section of the black hole would not have any horizon term. In this respect it makes sense to interpret it as a sort of classical entanglement of states. It is a very peculiar feature of gravity, because only gravity has the hamiltonian which on-shell is given by pure boundary terms.

The variation of H becomes

$$\delta H = -\left[\sigma^{-1}\eta'\delta N - \sigma^{-1}\eta\delta N' + \sigma^{-2}\eta N'\delta\sigma\right]_{x_0}^{X} \tag{2.12}$$

plus bulk terms giving the equations of motion. So again we should define

$$H_{\rm NEW} = H - \left(\sigma^{-1}\eta N'\right)_{|X} \tag{2.13}$$

[1]They were also discussed in thermodynamics ensembles by J.D. Brown and collaborators [20].

[2]Because we are going to discuss the canonical ensemble.

[3]A rather general treatment of phase space formulations of $2D$ gravity models can be found in [21], and references therein.

and this will be suitable to our boundary conditions. Notice that the $\delta\eta$ never appears on the horizon, so we need not a boundary condition for η at x_0. Finally

$$
\begin{aligned}
H_{\mathrm{NEW}} = \text{constraint term} - \left(\sigma^{-1}\eta N'\right)_{|x_0} + N(X)\left[(\sigma^{-1}\eta')_{|B} - (\sigma^{-1}\eta')_{|X}\right] + \\
+ \left(\sigma^{-1}\eta' N\right)_{|x_0}
\end{aligned}
\tag{2.14}
$$

the last term being really zero for a dilaton regular on the horizon. We also included a background subtraction, labelled by the subscript "B", to insure the convergence of the limits as $X \to \infty$. The natural ground state would be in the same topology class as the black hole and would have zero temperature. This identifies it as the metric (2.3) with $a = 0$. The horizon boundary term in H_{NEW} is an instance of a general result pertaining to spacetimes with degenerate foliations [22, 23].

We are now in position to describe the (reduced) phase space of the black hole (2.3). With a linear dilaton, the constraint implies $\sigma^2 = (\lambda^2 x^2 - a^2)^{-1}$, but leaves undetermined the lapse and sets the momentum variables to zero. However, the horizon regularity conditions (2.8) fix the behaviour of N near the horizon to have the form

$$
N^2 = \frac{2\kappa^2}{\lambda a}\left(x - \frac{a}{\lambda}\right) + \beta(t)\left(x - \frac{a}{\lambda}\right)^2 + O_3
\tag{2.15}
$$

and near infinity to be of order $\lambda^2 x^2$, but leaves N otherwise arbitrary. So, roughly speaking, there are as many metrics with a given surface gravity as there are analytic functions on the disk, vanishing in the origin and satisfying (2.15). Certainly this includes the full conformal group in $2D$, as this also is generated by analytic functions on the disk.

3. Thermodynamics and CFT

The boundary term at infinity in H_{NEW} can be easily evaluated, and leads to identify the mass as

$$
M = \frac{1}{2}\eta_0 \lambda a^2 \,.
\tag{3.1}
$$

The boundary term on the bifurcation point of the horizon is new, and has the effect to shift the mass to a lower value. In the spirit of CFT, we interpret then the hamiltonian as

$$
H_{\mathrm{NEW}} = \kappa\left(L_0 + \tilde{L}_0 - \frac{c + \tilde{c}}{24}\right).
\tag{3.2}
$$

The other Virasoro's generators L_n, $\tilde{L}_n$, would be just the on shell value of the hamiltonians $H[\xi_n]$, $H[\tilde{\xi}_n]$ for surface deformations [25] generated by vector fields respecting the horizon regularity conditions near the bifurcation point. We may use the time inversion symmetry of the solution to infer that, in fact, $c = \tilde{c}$. This stems from the fact that static black holes do have indeed a pair of classical Virasoro algebras connected with the canonical realizations of symmetries on the phase space [26, 27], but these are linked to the future and past sheet of the horizon, respectively.

Since the horizon term in H_{NEW} is $-\kappa\eta_{\mathrm{hor}} = -\kappa\eta_0 a$, this gives a central charge

$$c = 12\eta_0 a\,. \tag{3.3}$$

What's about L_0 and $\tilde{L}_0$? In a Lorentzian world there are in fact two horizons, but in the canonical euclidean picture there is only one, the axis of the euclidean section. It is natural to consider the sector of the CFT in which all Virasoro generators of the past horizon, $\tilde{L}_n$, vanish for $n \leq 0$.[4] This means imposing that white holes have no entropy. But then $L_0 = \eta_0 a/2$ and Cardy's formula gives the entropy as

$$S = 2\pi\sqrt{\frac{cL_0}{6}} = 2\pi\eta_0 a\,. \tag{3.4}$$

We may confirm this simply comparing $T_H dS$ with dM, where $T_H = (2\pi)^{-1}\lambda a$ is the Hawking's temperature of the black hole. Or we may evaluate the off-shell euclidean action (a la Gibbons-Hawking [24]), which is of independent interest, choosing as a reference background the zero temperature state (the extreme black hole metric with $a = 0$). Then we obtain

$$I_E = -\frac{1}{2}\int |g|^{1/2}\eta(R + 2\lambda^2) + \lim_{X\to\infty}\int_0^\beta d\tau\left[N\eta_B N_B' - \sigma^{-1}\eta N'\right]_{|X} \tag{3.5}$$

and we have also included the background subtraction. We recall that this has the same dilaton $\eta_B = \eta_0\lambda x$ and $N_B = \lambda x$. Using the Gauss-Bonnet theorem for a disk we obtain

$$I_E = -\frac{(2\pi)^2\eta_0}{\lambda\beta_H} + \frac{(2\pi)^2\eta_0}{2\lambda\beta_H^2}\,\beta\,, \tag{3.6}$$

where $\beta_H = 2\pi\lambda^{-1}a^{-1}$ is the Hawking's inverse temperature of the black hole (2.3) and β is arbitrary. Since $\log Z = -I_E$, this leads to a mass $M = 2^{-1}\eta_0\lambda a^2$ and entropy $S = 2\pi\eta_0 a$, as expected from the hamiltonian and the black hole first law, respectively.

The on shell action is I_E for $\beta = \beta_H$, that is

$$I_E \equiv -\log Z = -\frac{(2\pi)^2\eta_0}{2\lambda}\beta_H^{-1}\,. \tag{3.7}$$

This is the tipical behaviour of the partition function of a scale invariant theory in $2D$, and again gives M and S in agreement with Cardy's formula, solving also the $\sqrt{2}$ puzzle.

[4]The generators $\tilde{L}_n$, $n > 0$ vanish too, because they correspond to spacetime diffeomorphisms having a zero of order n in $(x - x_0)$.

The partition function of AdS black holes in higher dimensions also has the high temperature behaviour of scale invariant theories, $\log Z \sim C\beta^{-(D-1)}$ [28]–[32], but this is now understood as a consequence of duality with a gauge theory [33]. In AdS_2 the question of the dual field theory is harder, although recently a definite proposal has been made [8], which also solves the $\sqrt{2}$ puzzle. We do not rely, however, on microscopic details. The more puzzling remains the Schwarzschild black hole, which has $\log Z \sim \beta^2$, not that of a scale invariant theory, but that nevertheless has a horizon central charge[5] matching its entropy, if one uses Cardy's formula with the same assumptions made here on the Virasoro spectrum.

4. Conclusion

It is certainly amusing that thermodynamics and Cardy's formula match each other to such a high degree. What we find even more remarkable is that a knowledge of the classical hamiltonian is sufficient to determine the central charge of a quantum Virasoro algebra without appealing to any symmetry consideration. This result is not limited to two-dimensional models and will be further discussed in a separate paper. Due to these facts, we have confidence that our assumptions on the spectrum of the Virasoro algebra correctly describe $2D$ black holes. It also implies that white holes have no entropy. There is, however, a puzzling aspect for extreme black holes. The energy shift in the hamiltonian from the horizon, is always proportional to the surface gravity of the black hole, and this vanishes for extreme states. Hence the non-zero entropy of stringy extremal black holes, like the D1/D5 system [34, 35], remains an open question.

That the central charge appears in the classical hamiltonian for the outer region of the black hole, while it is absent for a complete Cauchy surface, can be interpreted as the classical limit of a sort of entanglement of states.

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